2/20/26 Return quizzes

Exam 1 next Wed 2/25.

Closed book - No notes.

Covers all of ch1, ch2. None of ch3. (Although we'll start Ch3 Wednesday. 2/18)

<u>No</u> calculators! Cell phones should be put away.

Practice exam + solutions are on webpage.

Last time:

Defined for a function $f: A \to B$,

$$\text{image}(f) = \{ b \in B \mid b = f(x) \text{ for some } x \in A \}$$

For the lin transf T def by $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ we saw

$$\left.\begin{aligned} \text{image}(T) &= \text{span}(\text{columns of } A) \\ &= \text{column space of } A \end{aligned}\right\} \text{ always true (for } \underline{\text{any}}\ A)$$

$$= \text{all of } \mathbb{R}^3 \quad (\text{only because } A \text{ is invertible})$$

<u>Ex</u> $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$\text{image}(T) = \text{column space of } A$$
$$= \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right)$$

scalar   vector

But $x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = (x_1 + 2x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So $\text{image}(T)$ is the line spanned by 0 and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

theorem (3.1.4)

If $T: \mathbb{R}^m \to \mathbb{R}^n$ is a lin. transf. then

(a) the vector $\vec{0}$ in $\mathbb{R}^n$ is in image(T)

(b) image(T) is closed under addition, ie if $\vec{v}, \vec{w} \in$ image(T) then $\vec{v} + \vec{w} \in$ image(T)

(c) image(T) is closed under scalar mult: if $\vec{v} \in$ image(T) and $k \in \mathbb{R}$ then $k\vec{v} \in$ image(T)

proof

(a) $T(\vec{0}) = \vec{0}$ so $\vec{0} \in$ image(T)

in $\mathbb{R}^m$ in $\mathbb{R}^n$

(b) If $\vec{v} = T(\vec{x}_1)$, $\vec{w} = T(x_2) \Rightarrow \vec{v} + \vec{w} = T(\vec{x}_1 + \vec{x}_2)$

(c) If $\vec{v} = T(\vec{x}_1) \Rightarrow k\vec{v} = T(k\vec{x}_1)$ //

Ex let A be an $n \times n$ matrix, and let T be def. by A and T^2 be def by A^2.

Claim image$(T^2) \subseteq$ image(T)

Say $\vec{v} \in$ image$(T^2) \Rightarrow \vec{v} = T^2(\vec{x})$ for some $x \in \mathbb{R}^n$ $\Rightarrow \vec{v} = T(T(\vec{x}))$ //

Ex Give an example of a lin transf $T: \mathbb{R}^3 \to \mathbb{R}^3$ whose image is the plane $3x + 4y + 5z = 0$.

we want the col. space of A to be this plane.

For example $\begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -5 \\ 4 \end{bmatrix}$ lie on the plane, and span it.

Take $T(\vec{e}_1) = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 0 \\ -5 \\ 4 \end{bmatrix}$, $T(\vec{e}_3) =$ sum $= \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}$

$$A = \begin{bmatrix} -4 & 0 & -4 \\ 3 & -5 & -2 \\ 0 & 4 & 4 \end{bmatrix}$$ works. (It's not the only possible solution.)

<u>The kernel of a lin. transf.</u>

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a lin. transf. Then

$$\ker(T) = \{ \vec{x} \in \mathbb{R}^m \mid T(\vec{x}) = \vec{0} \}$$

!! Equiv: if T is def by an $n \times m$ matrix A then

$$\ker(T) = \{ x \in \mathbb{R}^m \mid A\vec{x} = \vec{0} \}$$

$=$ solution set of the homog. linear system $A\vec{x} = \vec{0}$

We'll write ker(T) or ker(A) interchangeably.

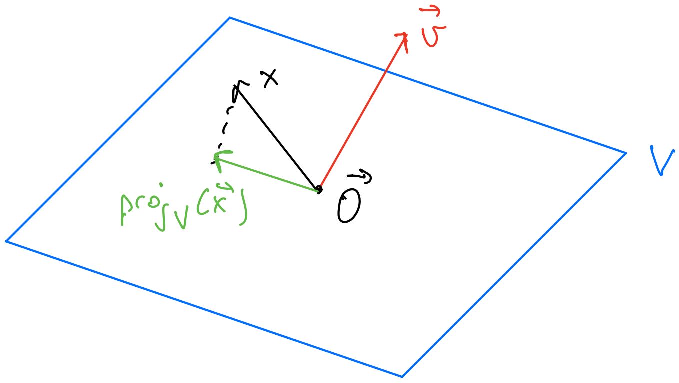
Ex Let $V \subset \mathbb{R}^3$ be a plane through $\vec{0}$ and let

$T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(\vec{x}) = \text{proj}_V(\vec{x})$.

orthogonal projection

Find ker(T) and image(T)

Let $\vec{v}$ be an orthogonal vector to V.



$\text{image}(T) = V$

$\ker(T) =$ line ℓ spanned by $\vec{v}$

Ex Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 9 & 6 \end{bmatrix}$ $T : \mathbb{R}^3 \to \mathbb{R}^2$. Find $\ker(T)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 9 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solve the lin syst

$$\begin{aligned} x + 2y + 3z &= 0 \\ 4x + 9y + 6z &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 9 & 6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 15 & 0 \\ 0 & 1 & -6 & 0 \end{array}\right]$$

$x = -15t$

$y = 6t$

$z = t$

So $\ker(T)$ is the subspace spanned by $\begin{bmatrix} -15 \\ 6 \\ 1 \end{bmatrix}$

See Ex 11 in §3.1 for a similar example